RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. SECOND SEMESTER EXAMINATION, MAY 2018

FIRST YEAR (BATCH 2017-20)

MATHEMATICS (Honours)

Date : 19/05/2018

Time : 11.00 am – 3.00 pm

Paper : II

Full Marks : 100

[Use a separate Answer Book for each Group]

Group - A

Answer <u>any five</u> questions from <u>Question No. 1 to 8</u>: (5 × 5)

1. If a, b, c be positive real numbers, not all equal, and n is a negative rational number, prove that

$$a^n(a-b)(a-c)+b^n(b-a)(b-c)+c^n(c-a)(c-b)>0$$

2. a) If $a_1, a_2, \cdots, a_n$ be n positive rational numbers and $s = a_1 + a_2 + \cdots + a_n$, prove that

$$\left(\frac{s}{a_1}-1\right)^{a_1}\left(\frac{s}{a_2}-1\right)^{a_2}\cdots\left(\frac{s}{a_n}-1\right)^{a_n} \le \; n-1^{\,s}.$$

b) Find the minimum value of $3x+2y$ when x, y are positive real numbers satisfying the condition $x^2y^3 = 48$. (3+2)

3. Verify that $Log(-i)^{1/2} = \frac{1}{2}Log(-i)$.

4. Prove that $x^n - 1 = (x^2-1)\prod_{k=1}^{\frac{n-2}{2}}\left[x^2 - 2x\cos\frac{2k\pi}{n}+1\right]$, if n be an even positive integer. Deduce that $\sin\frac{\pi}{32}\sin\frac{2\pi}{32}\sin\frac{3\pi}{32}\cdots\cdots\sin\frac{15\pi}{32} = \frac{1}{2^{13}}$. (4 + 1)

5. If α, β, γ are the roots of the equation $x^3+qx+r=0$, find the value of (i) $\sum\alpha^5$ and (ii) $\sum\frac{1}{\alpha^2-\beta\gamma}$. (3 + 2)

6. Prove that the equation $(x+1)^4 = a(x^4+1)$ is a reciprocal equation if $a \neq 1$ and solve it when $a=-2$. (2 + 3)

7. Find the special roots of the equation $x^{15}-1=0$. Deduce that $2\cos\frac{2\pi}{15}$, $2\cos\frac{4\pi}{15}$, $2\cos\frac{8\pi}{15}$, $2\cos\frac{16\pi}{15}$ are the roots of the equation $x^4-x^3-4x^2+4x+1=0$. (2 + 3)

8. Solve the equation: $x^4+2x^3-7x^2-8x+12=0$ by Ferrari's method.

Answer <u>any five</u> questions from <u>Question No. 9 to 16</u>: [5×5]

9. a) Prove or disprove: The range of any convergent sequence in $\mathbb{R}$ is a compact set.

b) Prove that every compact set in $\mathbb{R}$ is closed and bounded. (2+3)

10. a) Let $a_n \;{}_{n\in\mathbb{N}}$ be a sequence of real numbers. If $\sum_{n=1}^{\infty}|a_n|$ is convergent, then show that $\sum_{n=1}^{\infty}a_n^2$ is convergent. Does the converse of the above result always hold? Support your answer.

b) Test the convergence of the series $\sum_{n=1}^{\infty} a_n$, where $a_n = \begin{cases} \frac{1}{n^2} & \text{where } n \text{ is not a square integer} \\ \frac{1}{n^{3/2}} & \text{where } n \text{ is square integer} \end{cases}$. ((2+1)+2)

11. a) Test the convergence of the series $1+\frac{2^2}{3^2}x+\frac{2^2\cdot 4^2}{3^2\cdot 5^2}x^2+\cdots,\ x>0$.

b) If $s=\sum_{n=1}^{\infty}(-1)^n a_n \quad a_n>0\ \forall n$ where a_n is monotone decreasing and $a_n \to 0$ as $n\to\infty$. Then prove that $|s-s_n| \le a_{n+1} \forall n$ where s_n is the nth partial sum of the series. (3+2)

12. A function $f:\mathbb{R}\to\mathbb{R}$ is defined by

$$f(x)=2x\ ,\ x\in\mathbb{Q}$$
$$=1-x,\ x\in\mathbb{R}\setminus\mathbb{Q}$$

show that f is continuous at $1/3$ and discontinuous at every other point.

13. a) Let f_n be a sequence of real-valued continuous functions on [0,1]. Show that the map $g:[0,1]\to\mathbb{R}$ defined by $g(x)=\max_{1\le i\le n} f_i(x),\ 0\le x\le 1$, is continuous on [0,1].

b) If $f:[0,1]\to\mathbb{R}$ sends Cauchy sequences to Cauchy sequences, does it follow that f is uniformly continuous? (2+3)

14. Let $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ be continuous on $[a, b]$ and let $f(a)<g(a),\ f(b)>g(b)$. Show that there exists a point $c\in(a,b)$ such that $f(c)=g(c)$. Deduce that $\cos x = x^2$ for some $x\in\ 0, \pi/2$. (2+3)

15. Let $M>0$ be a real number and $f:[a,b]\to\mathbb{R}$ be a function such that $|f(x)-f(y)|<M|x-y|^5\ \forall x,y\in[a,b]$. Prove that f is constant on $[a, b]$.

16. a) Find the points of local maximum and local minimum of the function $f(x)=(x-1)^2(x-3)^3,\ x\in\mathbb{R}$.

b) If $a_1,a_2,\cdots,a_n$ are all positive real numbers, find the value of $\lim_{x\to\infty}\left[\frac{a_1^{\frac{1}{x}}+a_2^{\frac{1}{x}}+\cdots+a_n^{\frac{1}{x}}}{n}\right]^{nx}$. (2+3)

Group - B

***Answer any two questions from Question No. 17 to 19*:** [2×10]

17. a) If $\alpha_1,\alpha_2,\cdots,\alpha_n$ be a basis of a finite dimensional vector space V over a field F, then prove that any set of linearly independent vectors of V contains at most n vectors.

b) Prove that $\begin{vmatrix} (b+c)^2 & c^2 & b^2 \\ c^2 & (c+a)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix} = 2(bc+ca+ab)^3$.

c) A and B are real orthogonal matrices of same order and $\det A+\det B=0$. Show that A + B is a singular matrix. (5+3+2)

18. a) Obtain non-singular matrices P and Q such that $PAQ = R$ where R is the fully reduced normal form of A where $A=\begin{bmatrix} 1 & 0 & 2 & 2 \\ 2 & 1 & 5 & 0 \\ 3 & 1 & 9 & 3 \end{bmatrix}$.

b) Expand by Laplace's Method to prove that $\begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2$.

c) Find a basis for the vector space $\mathbb{R}^3$ that contains the vectors (2,4,0) and (2,6,2). (4+4+2)

19. a) Find the dimension of the subspace S of $\mathbb{R}^4$ defined by $S = \{(x,y,z,w) \in \mathbb{R}^4 : x+2y-z=0,\ 2x+y+w=0\}$.

b) Use elementary row operations on A to obtain A^{-1} where A is $\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$.

c) Let V be the vector space of all 2 × 2 matrices over the field F. Let $S=\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ be a subset of V. Prove that S is a basis of V. (4+3+3)

***Answer <u>any two</u> questions from <u>Question No. 20 to 22</u>*:** [2×12]

20. a) Prove that if for a B.F.S. $\boldsymbol{x_B}$ of a Linear programming problem

$$\text{Maximize } \boldsymbol{z} = \boldsymbol{cx}$$
$$\text{Subject to } \boldsymbol{Ax} = \boldsymbol{b},\ \boldsymbol{x} \geq 0$$

we have $z_j - c_j \geq 0$ for every column $\boldsymbol{a_j}$ of $\boldsymbol{A}$, then $\boldsymbol{x_B}$ is an optimal solution.

b) Identical products are produced in three factories and sent to four warehouses for delivery to the customers. The cost of transportation and capacities are given by the cost matrix as,

	W_1	W_2	W_3	W_4	a_i
F_1	3	8	7	4	30
F_2	5	2	9	5	50
F_3	4	3	6	2	80
b_j	20	60	55	40	

(i) Find the optimal schedule of delivery for minimization of cost of transportation.

(ii) Find idle capacity of warehouses.

(iii) Do you anticipate any alternative optimum solution for the problem? How can the same be identified? (7+(3 + 1 + 1))

21. a) Solve the following Linear programming problem by Big M-method:

$$\begin{aligned}
&\text{Minimize } z = 3x_1 + 5x_2 \\
&\text{Subject to } x_1 + 2x_2 \geq 8 \\
&3x_1 + 2x_2 \geq 12 \\
&5x_1 + 6x_2 \leq 60 \\
&x_1, x_2 \geq 0
\end{aligned}$$

b) Show that a B.F.S. to a Linear programming problem corresponds to an extreme point of the convex set of feasible solutions.

c) Prove that a hyperplane is a convex set. (5+4+3)

22. a) Write down the dual of the following problem and solving the dual problem by simplex method find the optimum solution of the primal problem and also optimum values of the primal and dual as well:

$$\begin{aligned}
&\text{Maximize } z = 3x_1 + 4x_2 \\
&\text{Subject to } x_1 + x_2 \leq 10 \\
&2x_1 + 3x_2 \leq 18 \\
&x_1 \leq 8 \\
&x_2 \leq 6 \\
&\text{and also } x_1, x_2 \geq 0
\end{aligned}$$

b) Prove that the number of basic variables in a transportation problem is atmost $m+n-1$.

c) Solve the following travelling salesman problem: ((2+3+1)+2+4)

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	1	2	3	4	5
1	—	6	12	6	4
2	6	—	10	5	4
3	8	7	—	11	3
4	5	4	11	—	5
5	5	2	7	8	—

***Answer <u>any one</u> question from <u>Question No. 23 to 24</u>*:** [1×6]

23. Use two phase method to show that the following L.P.P. has unbounded solution: 6

$$\begin{aligned}
&\text{Maximize } z = 2x_1 - x_2 + 2x_3 \\
&\text{Subject to } x_1 + x_2 - 3x_3 \leq 8 \\
&4x_1 - x_2 + x_3 \geq 2 \\
&2x_1 + 3x_2 - x_3 \geq 4 \\
&x_1, \quad x_2, \quad x_3 \geq 0
\end{aligned}$$

24. a) Prove that if a finite optimal feasible solution exists for the primal, then there exists a finite optimal feasible solution for the dual.

b) Define supporting hyperplane and separating hyperplane. (4+2)

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